4.4 - Undetermined Coefficients–Superposition Approach

The techniques in this lesson apply to nonhomogeneous DEs of the form $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x)$ whose input functions (that is, $g(x)$) are constants, polynomials, exponentials, sines and cosines, and sums and products of these.

From 4.1: Consider $y'' - 6y' + 5y = \underbrace{-9e^{2x}}_{g_1(x)} + \underbrace{5x^2 + 3x - 16}_{g_2(x)}$

the complete (general) solution is

$$y = \underbrace{c_1 e^x + c_2 e^{5x}}_{y_c \text{ complementary function}} + \underbrace{\underbrace{3e^{2x}}_{y_{p_1}} + \underbrace{x^2 + 3x}_{y_{p_2}}}_{y_p \text{ - particular solution}}$$

$g(x)$ comprises an exponential function and a polynomial

Starting with $y'' - 6y' + 5y = -9e^{2x}$

$g_1(x)$ input/forcing function

Aux eqn: $m^2 - 6m + 5 = 0$
$(m-5)(m-1) = 0$
$m = 1, 5$

Since $g_1(x)$ has the form of an exponential, we "guess" that

$y_{p_1} = Ae^{2x} \Rightarrow y_{p_1}' = 2Ae^{2x}, \quad y_{p_1}'' = 4Ae^{2x}$

$4Ae^{2x} - 12Ae^{2x} + 5Ae^{2x} = -9e^{2x}$

$4A - 12A + 5A = -9 \Rightarrow -3A = -9 \Rightarrow A = 3$

$y_{p_1} = 3e^{2x}$

Considering $g_2(x) = 5x^2 + 3x - 16$,

$y_{p_2} = Ax^2 + Bx + C$

$y_{p_2}' = 2Ax + B, \quad y_{p_2}'' = 2A$

$$2A - 12Ax - 6B + 5Ax^2 + 5Bx + 5C = 5x^2 + 3x - 16$$

$A = 1 \qquad -12 + 5B = 3 \Rightarrow B = 3$

$2 - 18 + 5C = -16 \Rightarrow C = 0$

$y_{p_2} = x^2 + 3x$

$y = y_c + y_p$

$\Rightarrow y = c_1 e^{5x} + c_2 e^{x} + 3e^{2x} + x^2 + 3x$

Solve the given differential equation by undetermined coefficients.

Ex: $y'' - 8y' + 20y = 100x^2 - 26xe^x$

Homog DE: $y'' - 8y' + 20y = 0$

$m^2 - 8m + 16 = -20 + 16$

$(m-4)^2 = -4 \Rightarrow m = 4 \pm 2i$

$$y_c = e^{4x}(c_1\cos 2x + c_2 \sin 2x)$$

option 1: $y_{p_1} = Ax^2+Bx+C$ $\qquad$ $\underline{y_{p_2} = (Dx+E)e^x}$

option 2: $y_p = Ax^2+Bx+C+(Dx+E)e^x$

option 1: we'd find that $y_{p_1} = 5x^2+4x+\frac{11}{10}$

$$y_{p_2}' = \underline{(D+Dx+E)e^x}$$

$$y_{p_2}'' = (D+D+Dx+E)e^x = \underline{(2D+Dx+E)e^x}$$

into $\underline{y''}-8\underline{y'}+20\underline{y} = -26xe^x$

we'd find that $y_{p_2} = (-2x-\frac{12}{13})e^x$

We found $y_c = e^{4x}(c_1\cos 2x + c_2\sin 2x)$,

$y_{p_1} = 5x^2+4x+\frac{11}{10}$, and

$y_{p_2} = (-2x-\frac{12}{13})e^x$

So

$$\boxed{y = e^{4x}(c_1\cos 2x + c_2\sin 2x) + 5x^2+4x+\frac{11}{10} + (-2x-\frac{12}{13})e^x}$$

$$(ue^{3x})' = (u'+3u)e^{3x}$$

Ex: $y'' - 16y = 2e^{4x}$

possible

Setting up for failure:

$$y_p = Ae^{4x} \Rightarrow y_p' = 4Ae^{4x}$$

$$y_p'' = 16Ae^{4x}$$

$$16Ae^{4x} - 16Ae^{4x} = 0$$

But $m^2 - 16 = 0 \Rightarrow m = \pm 4$

$$y_c = c_1 e^{4x} + c_2 e^{-4x}$$

so $y_p = Axe^{4x}$

include the minimum # of factors of x so we don't repeat any solutions in the complementary function

Ex: $4y'' - 4y' - 3y = \cos 2x$

y_p forms, assuming no redundancy

$g(x)$	y_p
$3x^2 + 2$	Ax^2+Bx+C
$7e^{4x}$	Ae^{4x}
$8x^2e^{2x}$	$(Ax^2+Bx+C)e^{2x}$
$3 \sin 2x$	$A \sin 2x + B \cos 2x$
$x \cos 3x$	$(Ax+B) \cos 3x + (Cx+D) \sin 3x$